Comparing the Fairness of Recursively Balanced Picking Sequences

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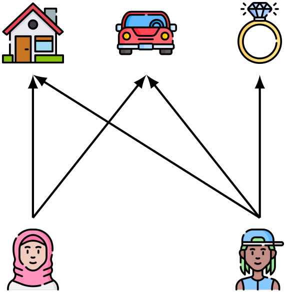
Fair Division

- How can we fairly allocate resources among multiple agents?
- Example: inheritance.



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Definitions

- An instance consists of:
  - A set of agents $N = \{1, \ldots, n\}$.
  - A set of indivisible goods $M = \{g_1, \ldots, g_m\}$.
  - Each agent $i \in N$ has a utility function $u_i : 2^M \to \mathbb{R}_{\geq 0}$, which is:
    - non-negative: $u_i(g) \geq 0$ for any $g \in M$; and
    - additive: $u_i(S) = \sum_{g \in S} u_i(g)$ for any $S \subseteq M$.
- An allocation $(A_1, \ldots, A_n)$ is an ordered partition of the goods M,
  - where each agent $i \in N$ is allocated bundle A_i, which may be empty.
- An allocation is envy-free up to one good (EF1) if for any agents $i, j \in N$, either $A_j = \varnothing$ or there exists a good $g \in A_j$ such that

$$u_i(A_i) \geq u_i(A_j \setminus \{g\}).$$

Picking Sequences

- Definition:
  - A sequence of agents $(a_1, a_2, \ldots, a_m)$.
  - In the j-th turn, agent a_j picks an available good with the highest utility according to her.
- Advantages:
  - Simplicity: Easy to implement in polynomial time.
  - Privacy: Agents only need to reveal their picks, not their utilities.
- A picking sequence guarantees envy-freeness up to one good (EF1) if and only if it is recursively balanced, i.e.,
  - it can be divided into rounds where every agent picks once in each round,
  - except in the last (possibly incomplete) round where each agent picks at most once.
- Examples of recursively balanced picking sequences:
  - Round-robin: $(1, 2, \ldots, n \mid 1, 2, \ldots, n \mid 1, 2, \ldots, n)$
  - Balanced alternation: $(1, 2, \ldots, n \mid n, \ldots, 2, 1 \mid 1, 2, \ldots, n)$

Example

	Good g_1	Good g_2	Good g_3	Good g_4
Agent 1	4	3	2	1
Agent 2	4	3	2	1

Round-robin: $(1, 2 \mid 1, 2)$

- Agent 1 gets a utility of 6. Agent 2 gets a utility of 4.
- Egalitarian welfare $= \min\{6, 4\} = 4$.

	Good g_1	Good g_2	Good g_3	Good g_4
Agent 1	4	3	2	1
Agent 2	4	3	2	1

Balanced alternation: $(1, 2 \mid 2, 1)$

- Agent 1 gets a utility of 5. Agent 2 gets a utility of 5.
- Egalitarian welfare $= \min\{5, 5\} = 5$.

Question: Which recursively balanced picking sequence is the "fairest"?

Our Contribution

Assumption: Without loss of generality, assume that a picking sequence starts with $1, 2, \ldots, n$.

We compare all recursively balanced picking sequences using two fairness benchmarks:

- Egalitarian price
- Maximin share (MMS) guarantee

Egalitarian Price

Definition: The egalitarian price of a picking sequence π is

$$\underset{\text{across all instances}}{\text{supremum}} \frac{\text{optimal egalitarian welfare}}{\text{egalitarian welfare obtained using } \pi}.$$

Results: The egalitarian price is the same for all recursively balanced picking sequences, whether the optimal egalitarian welfare is defined across

- all allocations: ∞,
- all picking sequences: $\min\{n, m-n+1\}$, or
- all recursively balanced picking sequences: $\min\{\lfloor \log_2 n \rfloor + 1, \lceil m/n \rceil\}$.

Maximin Share (MMS) Guarantee

Definition:

- An agent's maximin share (MMS) is the maximum utility she can get if:
  - the goods are partitioned into n bundles and
  - she is allocated a bundle with the lowest utility.
- A picking sequence's MMS guarantee is the largest α such that every agent is guaranteed to get at least α times her MMS.

Maximin Share (MMS) Guarantee (Cont'd)

Results:

- We provide a tight MMS guarantee for each recursively balanced picking sequence.
  - If $m \geq 2n$, the MMS guarantee depends only on the positions of agent n in the picking sequence.
  - If $m < 2n$, the MMS guarantee depends only on the positions of agents n and $n-1$ in the picking sequence.

Maximin Share (MMS) Guarantee (Cont'd)

- We characterize the best and worst recursively balanced sequences: Let $m \geq 2n$.* A recursively balanced picking sequence has
  - the worst MMS guarantee if and only if
    - in some round $r \geq 2$ that is complete, agent n picks last or
    - the last round has exactly $n-1$ turns and agent n gets no turn.

    E.g.:
    - Round-robin: $(1, 2, \ldots, n \mid 1, 2, \ldots, n \mid 1, 2, \ldots, n)$
    - Balanced alternation: $(1, 2, \ldots, n \mid n, \ldots, 2, 1 \mid 1, 2, \ldots, n)$
  - the best MMS guarantee if and only if in every round $r \geq 2$, the overall index of agent n's pick is at most

$$n + (r-1)\alpha_{\max},$$

    where

$$\alpha_{\max} = \min\left\{ \frac{\lfloor m/n \rfloor}{\lfloor m/n \rfloor n - n + 1}, \frac{\lceil m/n \rceil}{m - n + 1} \right\}.$$

    - E.g: $(1, 2, \ldots, n \mid n, \ldots, 2, 1 \mid n, \ldots, 2, 1)$

*The characterizations differ when $m < 2n$.

Maximin Share (MMS) Guarantee (Cont'd)

Example: Let $n = 3$ and $m = 7$.

- Worst picking sequences:

$$(1, 2, \mathbf{3} \mid 1, 2, \mathbf{3} \mid 1) \qquad (1, 2, \mathbf{3} \mid 1, 2, \mathbf{3} \mid 2) \qquad (1, 2, \mathbf{3} \mid 1, 2, \mathbf{3} \mid 3)$$
$$(1, 2, \mathbf{3} \mid 2, 1, \mathbf{3} \mid 1) \qquad (1, 2, \mathbf{3} \mid 2, 1, \mathbf{3} \mid 2) \qquad (1, 2, \mathbf{3} \mid 2, 1, \mathbf{3} \mid 3)$$

- Best picking sequences:

$$(1, 2, \mathbf{3} \mid \mathbf{3}, 1, 2 \mid \mathbf{3}) \qquad (1, 2, \mathbf{3} \mid 1, \mathbf{3}, 2 \mid \mathbf{3})$$
$$(1, 2, \mathbf{3} \mid \mathbf{3}, 2, 1 \mid \mathbf{3}) \qquad (1, 2, \mathbf{3} \mid 2, \mathbf{3}, 1 \mid \mathbf{3})$$

- Remaining picking sequences:

$$(1, 2, \mathbf{3} \mid \mathbf{3}, 1, 2 \mid \mathbf{1}) \qquad (1, 2, \mathbf{3} \mid 1, \mathbf{3}, 2 \mid \mathbf{1})$$
$$(1, 2, \mathbf{3} \mid \mathbf{3}, 2, 1 \mid \mathbf{1}) \qquad (1, 2, \mathbf{3} \mid 2, \mathbf{3}, 1 \mid \mathbf{1})$$
$$(1, 2, \mathbf{3} \mid \mathbf{3}, 1, 2 \mid \mathbf{2}) \qquad (1, 2, \mathbf{3} \mid 1, \mathbf{3}, 2 \mid \mathbf{2})$$
$$(1, 2, \mathbf{3} \mid \mathbf{3}, 2, 1 \mid \mathbf{2}) \qquad (1, 2, \mathbf{3} \mid 2, \mathbf{3}, 1 \mid \mathbf{2})$$

Future Directions

- Average-case analysis, e.g.,
  Bouveret, S., & Lang, J. (2011). A general elicitation-free protocol for allocating indivisible goods. *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI)*, 73–78

- Non-recursively balanced picking sequences