Egalitarian Price of Fairness for Indivisible Goods

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Fair Division

- Resources are divided among a group of people.
- How can each person get a fair share?
- Example: inheritance.

Definitions

An instance consists of:

- Agents $1, \ldots, n$.
- Goods $1, \ldots, m$ which are indivisible.
- Each agent i values:
  - good g at $v_i(g) \geq 0$.
  - bundle of goods $v_i(\{g_1, \ldots, g_k\}) = v_i(g_1) + \cdots + v_i(g_k)$.
  - bundle of all goods at $v_i(1) + \cdots + v_i(m) = 1$

Fairness

Suppose each agent i is allocated a bundle of goods A_i.

- Envy-free: for all agents i, j,

$$v_i(A_i) \geq v_i(A_j)$$

- Envy-free up to 1 good (EF1): for all i, j, there is a $g \in A_j$ s.t.

$$v_i(A_i) \geq v_i(A_j \setminus \{g\})$$

- Balanced: for all agents i, j,

$$|A_i| \geq |A_j| - 1$$

- Round robin (RR): let the agents take turn choosing their most preferred good from the unallocated items.

Note: RR $\implies$ Balanced $\wedge$ EF1

Efficiency

Suppose each agent i is allocated a bundle of goods A_i.

We want to maximize the social welfare:

- Utilitarian welfare (UW): the sum

$$v_1(A_1) + \cdots + v_n(A_n)$$

- Nash welfare (NW): the product

$$v_1(A_1) \times \cdots \times v_n(A_n)$$

- Egalitarian welfare (EW): the minimum

$$\min \Big(v_1(A_1), \ldots, v_n(A_n) \Big)$$

Instance

Agent / Good	1	2	3
1	**3/8**	2/8	3/8
2	2/8	**3/8**	3/8
3	0	**4/8**	**4/8**

Alloc 1: Agent 1 gets good 1. Agent 2 gets good 2. Agent 3 gets good 3.

- Fair: balanced
- Efficient: maximizes egalitarian welfare (which is 3/8)

Alloc 2: Agent 1 gets good 1. Agent 3 gets goods 2 and 3.

- Efficient: maximizes utilitarian welfare (which is 11/8)

Sometimes there are conflicts between fairness and efficiency...
How much efficiency must we sacrifice to obtain fairness?

Price of Fairness

For property P and instance I

$$\text{POF}(P, I) = \frac{\text{Maximum welfare overall}}{\text{Maximum welfare satisfying } P}$$

For property P

$$\text{POF}(P) = \sup_I \text{POF}(P, I)$$

How do we define "welfare"?

Instance

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Alloc 2: Agent 1 gets good 1. Agent 3 gets goods 2 and 3.

- Efficient: maximizes utilitarian welfare (which is 11/8)

Utilitarian POF of balanced $= \frac{11/8}{10/8} = 11/10$.

Egalitarian POF of balanced $= \frac{3/8}{3/8} = 1$.

Results

Let n be the number of agents.

Property		POF	
		Egalitarian	Utilitarian[1,2]
EF1		$\Theta(n)$	$\Theta(\sqrt{n})$
Balanced		n	$\Theta(\sqrt{n})$
RR		$\Theta(n)$	n
MNW	$(n = 2)$	≈ 2	≈ 1.2
	$(n \geq 3)$	∞	$\Theta(n)$
MUW		∞	1
MEW		1	$\Theta(n)$

[1] Barman, Bhaskar, and Shah 2020

[2] Bei et al. 2019

Balanced: POF $\geq n$

Agent/Good	1	$2, \ldots, m$
1	1	0
$2, \ldots, n-1$	$1-(m-1)\varepsilon$	ε
n	$1-(m-1)\varepsilon^2$	ε^2

Agent 1 gets good 1.

Opt allocation: Each agent gets one good. Agent n gets the rest.
Egalitarian Welfare $= (m-(n-1)) \cdot \varepsilon^2$

Balanced allocation: All agents split equally.
Egalitarian Welfare $= \frac{m}{n} \cdot \varepsilon^2$

$$\text{POF} \geq \frac{(m-(n-1)) \cdot \varepsilon^2}{\frac{m}{n} \cdot \varepsilon^2} \to n$$

Balanced: POF $\leq n$

Given an instance with n agents and m goods...

- Take an allocation with maximum egalitarian welfare (MEW).
- Make it balanced:
  - Agent has too many goods: take the excess goods.
    Let them keep their best m/n goods.
  - Agent has too few goods: give the excess goods.
- Each agent gets $\geq \frac{m/n}{m} = \frac{1}{n}$ of their MEW allocation.

$$\text{POF} \leq \frac{\text{MEW}}{\frac{1}{n} \cdot \text{MEW}} = n$$

Summary

Property		POF	
		Egalitarian	Utilitarian[3,4]
EF1		$\Theta(n)$	$\Theta(\sqrt{n})$
Balanced		n	$\Theta(\sqrt{n})$
RR		$\Theta(n)$	n
MNW	$(n = 2)$	≈ 2	≈ 1.2
	$(n \geq 3)$	∞	$\Theta(n)$
MUW		∞	1
MEW		1	$\Theta(n)$

- Egalitarian welfare is fairer than utilitarian welfare.
- Egalitarian POF is higher ~~lower~~ than utilitarian POF.

[3]Barman, Bhaskar, and Shah 2020

[4]Bei et al. 2019

Future work

- Other combinations of settings
  e.g. Nash welfare, chores, etc.
- Strong price of fairness which is defined by
  (Maximum welfare) / (Minimum ~~Maximum~~ welfare satisfying P).