On the Fairness of Additive Welfarist Rules

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Fair Division

- We allocate indivisible resources among a group of agents.
- How to ensure fairness and efficiency?
- Example: inheritance.

Definitions

- An instance consists of:
  - A set N of agents $1, \dots, n$.
  - A set M of goods $g_1, \dots, g_m$ which are indivisible.
  - Each agent i values:
    - good g non-negatively at $u_i(g) \geq 0$;
    - bundle of goods additively i.e. $u_i(\{g_1, \dots, g_k\}) = u_i(g_1) + \dots + u_i(g_k)$.
- An allocation $(A_1, \dots, A_n)$ is an ordered partition of goods M.
- Fairness:
  - Envy-free up to 1 good (EF1): For all i, j, either $A_j = \emptyset$ or there is a $g \in A_j$ with
$$u_i(A_i) \geq u_i(A_j \setminus \{g\}).$$
- Efficiency:
  - Additive welfarist rule with function f (which is strictly increasing): Chooses an allocation that maximizes
$$\sum_{i=1}^{n} f(u_i(A_i)).$$
  - Examples:
    - Maximum utilitarian welfare (MUW): $f(x) = x$.
    - Maximum Nash welfare (MNW): $f(x) = \log x$.
    - Maximum harmonic welfare (MHW): $f(x) = \Sigma_{k=1}^{x} 1/k$.

Past Work

- Real-valued:
  - MNW guarantees EF1 [Caragiannis et al., 2019].
  - MNW is the only additive welfarist rule which guarantees EF1 for all real-valued instances [Suksompong, 2023].
  - Additive welfarist rules based on p-mean functions with $p \leq 0$ also yield EF1 for normalized instances with two agents [Eckart et al., 2024].
- Integer-valued:
  - An important subclass in terms of real world application.
    - Example: Spliddit (http://www.spliddit.org/).
  - MHW guarantees EF1 for all integer-valued instances [Montanari et al., 2024].

Research Question

For each subclass of instances (e.g., integer-valued, two-value, normalized), which additive welfarist rules guarantee EF1?

Motivating Example

	g_1	g_2	g_3	$\cdots$	g_n
Agent 1	n				
Agent 2	$n{-}1$	1			
Agent 3		$n{-}1$	1		
$\vdots$			$\ddots$	$\ddots$	
Agent n				$n{-}1$	1

	g_1	g_2	g_3	$\cdots$	g_n
Agent 1	n				
Agent 2	$n{-}1$	1			
Agent 3		$n{-}1$	1		
$\vdots$			$\ddots$	$\ddots$	
Agent n				$n{-}1$	1

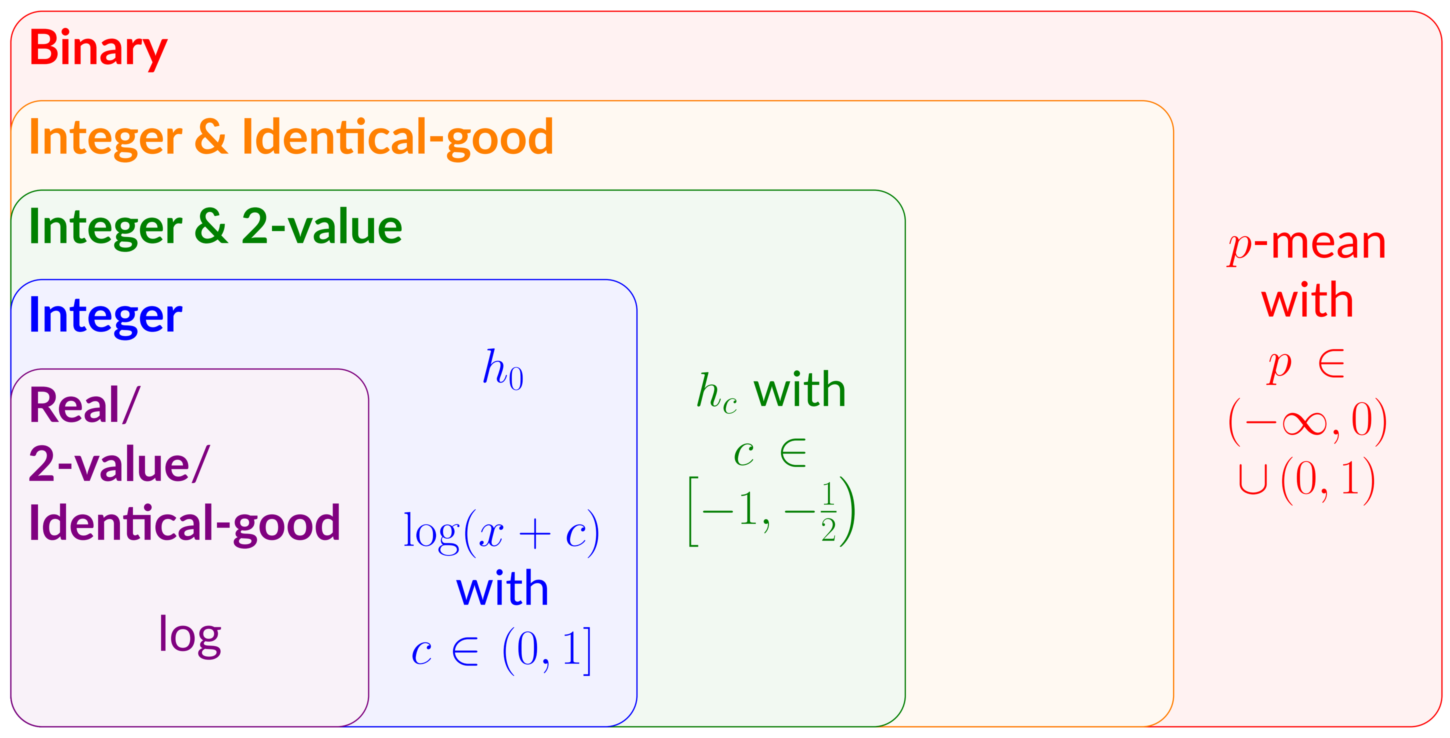
MNW: Agent 1 gets value n and every other agent gets value 1.

MHW: Agent 2 gets value 0 and every other agent gets value $\geq n{-}1$.

MHW allocation is arguably better than MNW allocation, especially when n is large.

Our Contribution

- Real-valued:
  - No restriction: MNW is the only additive welfarist rule guaranteeing EF1 [Caragiannis et al., 2019; Suksompong, 2023].
  - We extend this result to more restricted subclasses:
    - identical-good instances,
    - two-value instances, and
    - normalized instances with at least three agents.
- Integer-valued:
  - There are infinitely many other additive welfarist rules guaranteeing EF1.
    - Example: $f(x) = \log(x + c)$ for any $c \in [0, 1]$.
  - We give necessary and sufficient conditions.



Note: $h_c(x) = \Sigma_{t=1}^{x} \frac{1}{t+c}$ is the "modified harmonic number".

Future Work

- Other settings:
  - Non-additive welfarist rules
  - Other fairness notions (e.g. PROP1)
- Comparison:
  - For some subclasses (e.g. integer-valued instances), there are many additive welfarist rules that yield EF1.
  - Which rule is the "best"?
  - Method of comparison: e.g. price of fairness [Bei et al., 2021; Celine et al., 2023; Li et al., 2024].
- Computation: How to find allocations satisfying additive welfarist rules other than MNW?

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